Inverse and direct theorems for monotone approximation

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Abstract

We prove that if f is increasing function on $[-1,1]$ then for each $n = 1,2,\ldots$, there is an increasing algebraic polynomial p_n of degree $8n$ such that $\|f - p_n\|_p \le c(p)\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_p$,

Where ω_2^{φ} is the second order Ditizian – Totik modulus of smoothness. Also a converse theorem for this direct theorem were obtained. These results complement the classical pointwise estimates of the same type for unconstrained polynomial approximation.

1 .Introduction and Main Results

Several results show that in some sense monotone approximation by algebraic polynomials performs as well as unconstrained approximation. For example Lorentz and Zeller, [7] have shown that for each increasing function f in $C(I)$ (the space of all continuous functions on $I=[-1,1]$) there is an increasing polynomial p_n of degree n that satisfies

$$\|f-p_n\| \le c\ \omega\left(f,\frac{1}{n}\right), \quad n=1,2,... \qquad (1.1)$$

where ω is the modulus of continuity of f.

A general result for (2.1.1) for any $k=0,1,$ there are increasing p_n that satisfies

$$\|f-p_n\| \le c\,n^{-k}\ \omega\left(f^{(k)},\frac{1}{n}\right), \quad n=1,2,... \qquad (1.2)$$

this result of Lorentz [6], where as the general case was proved by DeVore [3], the cases $k=0,1$ are much easier to prove than the general cases. Since they can be proved using linear method, in contrast, the proof in [3] uses rather involved non linear techniques. It is well known that for unconstrained approximation much improvement can be made in estimates of the form (1.1) where x is near the end points of I.

In this thesis, we are interested in pointwise estimates for monotone approximation, the only result of this type that we know of is by Beatson [1]. He proved that the estimate

$$|f(x)-p_n(x)| \le c\omega\ (f,\Delta_n(x)), x\in I, n=1,2,...$$

$\Delta_n(x)=\sqrt{\frac{1-x^2}{n}}+\frac{1}{n^2}$, holds for suitable increasing polynomials p_n whenever f is increasing. Devore and Yn have shown that if f

is increasing function on $I=[-1,1]$, then for each $n=1,2,\ldots$ there is an increasing polynomial p_n of degree n such that

$$|f(x)-p_n(x)|\le c\omega_2\left(f,\sqrt{\frac{1-x^2}{n}}\right),$$

where ω_2 is the second order moduli of smoothness. Among other things, we shall show that this can be improved to allow second order modulus of smoothness for the spaces $L_p, 0<p<1$.

Theorem I: If f is an increasing function in $L_p(I), 0<p<1$ then for each $n=1,2,\ldots$ there is an increasing polynomial in $L_p(I)$, of degree (8n) satisfying

$$\|f-p_n\|_p \le c(p)\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_p. \tag{1.3}$$

Using this theorem we can obtain our second Inverse inequality:

Theorem II: Let f be an increasing function in $L_p(I), 0<p<1$, then

$$\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p \le c(p)E_n^1(f)_p^p + c(p)n^{-2p}\sum_{m>n}(m+1)^{p-1}E_n^1(f)_p^p.$$

2 .Auxiliary Lemmas

Before we prove our theorems we need the following notions and lemmas. Our proof is based on a two stage approximation. We first approximate f by an increasing piecewise linear function S_n. We then approximate S_n by an increasing algebraic polynomial . S_n is the piecewise linear function that interpolates f at $\xi_k, k=-n,...,n$, if we let s_j be the slops

$$s_j = \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j}, \quad j=-n,...,n-1 \quad . \tag{2.1}$$

Then S_n can be represented by using the function $\Phi_j(x) = \max\{(x-\xi_j),0\}$ as

$$S_n(x) = f(-1) + s_{-n}(x+1) + \sum_{j=n+1}^{n-1} (s_j - s_{j-1})\Phi_j(x), \text{ [4]} \tag{2.2}$$

It is clear that S_n is increasing if f is also.

We shall now contract a polynomial R_j, $j=-n,...,n-1$, as in [4] that approximate the function $\Phi_j(x)$. The construction of R_j begins with trigonometric polynomial $T_j, j=1,...,2n$ with

$t_j = \frac{j\pi}{2n}$, $j = 0,1,\ldots,2n$. Let K_n denote the Jackson kernel

$$K_n(t) = a_n \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^8, \qquad (2.3)$$

where a_n is a constant depending on n chosen so that

$$\int_{-\pi}^{\pi} K_n(t)dt = 1.$$

Here and throughout $c(p)$, c, denote absolute constants depending on p and $c(p), c$'s, values may vary with each occurrence on the same line.

Define

$$T_j(t) = \int_{t-t_j}^{t+t_j} K_n(u)du, j = 0,1,\ldots,2n, [4]$$

and define

$$d_j(t) = \max(n \ dist \ (t, \{-t_j, t_j\}), 1), [4] \qquad (2.4)$$

Now let

$$r_j(x) = T_{m-j}(t), \qquad x = \cos t.$$

And for $x \in [-1,1]$ define

$$R_j(x) = \int_{-1}^{x} r_j(u)du, j = -n,\ldots,n, [4] \qquad (2.5)$$

In particular $R_{-n}(x) = x + 1 = \Phi(x)$ and $R_n(x) = 0$, the points ξ_j are defined by the equations

$1 - \xi_j = R_j(1), j = -n,\ldots,n$. If $f \in L_p(I)$ we define

$$L_n(f) = f(-1) + s_{-n}R_{-n} + \sum_{-n+1}^{n-1}(s_j - s_{j-1})R_j. \quad (2.6)$$

with s_j defined by (2.1) if f is increasing, the $s_j \geq 0$, $j = -n,\dots,n-1$ and since we can also write

$$L_n(f) = f(-1) + \sum_{-n}^{n-1} s_j(R_j - R_{j+1}).$$

Now from the definition of the polynomials T_j we have $T_{n-j} - T_{n-(j+1)} \geq 0$, hence $r_j - r_{j+1} \geq 0$, and therefore $R_j - R_{j+1}$ is increasing , it follows that $L_n(f)$ is increasing.

We now estimate

$$E(x) = S_n(x) - L_n(f,x) = \sum_j (s_j - s_{j-1})(\Phi_j(x) - R_j(x)). \quad (2.7)$$

Now for $j = -n,\dots,n-1, x = \cos t$ with $0 \leq t \leq \pi$, we have

$$|\Phi_j(x) - R_j(x)| \leq cn^{-1}(\sin t_{n-j} + |t - t_{n-j}|)(d_{n-j}(t))^{-5} \text{ ,[20]} \quad (2.8)$$

Lemma 2.9. $\|L_n(f)\|_p \leq c(p)\|f\|_p$

Proof: We have

$$S_n(x) - L_n(f,x) = \sum_{j=-n+1}^{n-1}(s_j - s_{j-1})(\Phi_j(x) - R_j(x)).$$

Then

$$|L_n(f,x)| = \left|S_n(x) - \sum_{j=-n+1}^{n-1}(s_j - s_{j-1})(\Phi_j(x) - R_j(x))\right|$$

$$\leq |S_n(x)| + \left|\sum_{j=-n+1}^{n-1}(s_j - s_{j-1})(\Phi_j(x) - R_j(x))\right|.$$

Definition of S_n implies

$$\left|S_n(x)\right| \le \left|f(-1)\right| + \left|s_{-n}(x+1)\right| + \sum_{j=-n+1}^{n-1} \left|s_j - s_{j-1}\right| \left|\Phi_j(x)\right|.$$

Thus

$$\left|S_n\right| \le c\left|f(x)\right|.$$

And

$$\left|L_n(f,x)\right| \le c\left|f(x)\right| + \sum_{j=-n+1}^{n-1} \left|s_j - s_{j-1}\right| \left|\Phi_j - R_j\right|.$$

Then (2.2.1) implies

$\left|s_j\right| \le \dfrac{c\left|f(x)\right|}{\delta_j} \le \dfrac{c\left|f(x)\right|}{\delta_{j+1}}$ where $\delta_j = \xi_{j+1} - \xi_j$.

Then

$$\left|L_n(f,x)\right| \le c\left|f(x)\right| + c\left|f(x)\right| \sum_{j=-n+1}^{n-1} \delta_j^{-1} \left|\Phi_j - R_j\right|,$$

and by $\delta_j = \xi_{j+1} - \xi_j = \dfrac{c}{n}$, we have $\delta_j^{-1} \le cn$.

So

$$\left|L_n(f,x)\right| \le c\left|f(x)\right| + c\left|f(x)\right| \sum_{j=-n+1}^{n-1} \left(1 + n\left|t - t_{n-j}\right|\right)\left(d_{n-j}(t)\right)^{-5}$$

$$\le c\left|f(x)\right| + c\left|f(x)\right| \sum_{j=-n+1}^{n-1} \frac{1}{n^4}$$

Then by the following [4]

Lemma 2.10 If g' is absolutely continuous and $\left|g''\right| \le M$ almost every where on I. Then for each $n = 1,2,..$ and each $x \in I$, we have

$$\left|g(x) - L_n(g,x)\right| \le cM\left(\frac{\sqrt{1-x^2}}{n}\right)^2.$$

We can prove

Lemma 2.11 If g' is absolutely continuous and $|g''| \le M$ almost every where on I, then for each $n = 1,2,..,$ and each $x \in I$ we have

$$\|g - L_n(g)\|_p \le \frac{c(p)}{n^2}.$$

3 .proof of theorem I

Firstly let us introduce the so called Ditzian Totik functional definition as

$$\widetilde{K}_{2,\varphi}\left(f, \frac{1}{n^2}\right)_p = \inf_g\left(\|f - g\|_p + \frac{1}{n^2}\|\varphi^2 g''\|_p\right),$$

for $f \in L_p(I).0 < p \le \infty$.

We have

$$\omega_2^{\varphi}\left(f, n^{-1}\right)_p \approx \widetilde{K}_{2,\varphi}\left(f, \frac{1}{n^2}\right)_p \text{ [2].}$$

Given $x \in I$, then from the result above there is a g satisfies

$$\|f - g\|_p \le c(p)\omega_2^{\varphi}\left(f, n^{-1}\right)_p$$

and

$$\frac{1}{n^2}\|\varphi^2 g''\| \le c(p)\omega_2^{\varphi}\left(f, n^{-1}\right)_p \qquad (3.1)$$

$$\|f - L_n(f)\|_p^p \le \|f - g\|_p^p + \|g - L_n(g)\|_p^p + \|L_n(g) - L_n(f)\|_p^p.$$

Then by the linearity, and the boundedness of $L_n(f)$, we obtain

$$\|f - L_n(f)\|_p^p \le \|f - g\|_p^p + \|g - L_n(g)\|_p^p + \|L_n\|_p^p\|f - g\|_p^p$$

$$\leq \left(1+\|L_n\|_p^p\right)\|f-g\|_p^p+\|g-L_n(g)\|_p^p .$$

Lemma (2.11) implies $\|g-L_n(g)\|_p^p \leq \dfrac{c(p)}{n^2}$.

Using (3.1), Lemma(2.11) and the linearity of $L_n(f)$, we have

$$\|f-L_n(f)\|_p^p \leq \left(1+\|L_n\|_p^p\right)c(p)\,\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p+\frac{c(p)}{n^2}\|\varphi^2 g''\|_p^p$$

$$\leq \left(1+\|L_n\|_p^p\right)c(p)\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p+c(p)\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p .$$

By virtue of Lemma (2.9) we have

$$\|f-L_n(f)\|_p^p \leq c(p)\left(\left(1+\|f\|_p^p\right)\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p+\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p\right).$$

$$\leq c(p)\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p$$

Since $L_n(f)$ is an increasing polynomial of degree ≤ 8n we have proved theorem I ♣

4 .proof of Theorem II

For given by $= \max\{i:2^i<n\},2 = n$, we expand $p_n(x)$ by

$$p_n(x)-p_0(x)=\left(p_n(x)-p_2(x)\right)+\left(p_2(x)-p_{2^{-1}}(x)\right)+\quad+\left(p_1(x)-p_0(x)\right).$$

We recall that for $m<n$ $\quad \|p_n-p_m\|_p^p \leq c(p)E_m^1(f)_p^p$

$$\omega_2^{\varphi}\left(f,n^{-1}\right)_p^p \leq c(p)\|f-p_n\|_p^p+c(p)n^{-p}\ \omega_1^{\varphi}\left(p_n,n^{-1}\right)_p^p$$

$$\leq c(p)E_n^1(f)_p^p+c(p)n^{-2p}\|p_n'\|_p^p$$

$$\leq c(p)E_n^1(f)_p^p+c(p)n^{-2p}\left\|\sum_{i=1} p'_{2^i}\right\|_p^p,$$

where p_{2^i} is an algebraic polynomial of best monotone approximation of degree not greater than or equal 2 it mean

$$\left\| f - p_{2^i} \right\|_p^p = E_{2^i}^1(f)_p^p \ . \tag{4.1}$$

Then

$$\omega_2^{\varphi}\left(f, n^{-1}\right)_p^p \le c(p)E_n^1(f)_p^p + c(p)n^{-2p}\left\| \sum_{i=1} p'_{2^i} - p'_{2^{i-1}} \right\|_p^p$$

$$\le c(p)E_n^1(f)_p^p + c(p)n^{-2p}\sum_{i=1}\left\| p'_{2^i} - p'_{2^{i-1}} \right\|_p^p .$$

Then by Bernstein inequality we have

$$\omega_2^{\varphi}\left(f, n^{-1}\right)_p^p \le c(p)E_n^1(f)_p^p + c(p)n^{-2p}\sum_{i=1}^{\infty}\left(2^i n\right)^p \ E_{2^i n}^1(f)_p^p .$$

Applying the inequality

$$2^{ivp} \le c(p,v)\sum_{m=2^{i-1}+1}(m+1)^{vp-1}, v \in N \quad [5]$$

We get

$$\omega_2^{\varphi}\left(f, n^{-1}\right)_p^p \le c(p)E_n^1(f)_p^p + c(p)n^{-2p}\sum_{m>n}^{\infty}(m+1)^{p-1}E_m^1(f)_p^p \ \clubsuit$$

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